

Optimization technique for production-distribution planning in supply chain management under intuitionistic fuzzy environment

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Abstract A technique to find optimal solutions for production-distribution planning in supply chain management under intuitionistic fuzzy environment is presented in this paper. The advantages of using intuitionistic fuzzy sets over fuzzy sets or crisp sets under imprecise environment are well established. In 1997, Angelov proposed optimization technique under intuitionistic fuzzy environment. In 2009, Jimenez and Bilbao showed that fuzzy efficient solution may not be Pareto-optimal solution to multiple objective linear programming problems in case that one of the fuzzy goals is fully achieved. In 2015, Wu, Liu and Lur redefined membership function of fuzzy set theory and proposed another two phase technique. The prime intention to maximize the up-gradation of most misfortunate in optimization technique under intuitionistic fuzzy environment is better served when some constraints present in existing, well established techniques are removed. Moreover, usage of some such constraints present in existing techniques may make a problem infeasible. In proposed algorithm new functions are suggested in place of membership functions and non-membership functions of intuitionistic fuzzy sets. One standard production-distribution model is taken not only to allocate limited available resources and equipment to produce the products over time periods but also to determine economic distributors for dispatching product to the retailers. Numerical examples involving this production-distribution model further illustrate proposed techniques. Conclusions are drawn at last.

Keywords Intuitionistic fuzzy optimization · Production-distribution planning · $T^{(+)}$ -characteristic function · $T^{(-)}$ -characteristic function.

1 Introduction

Deterministic optimization techniques are well studied and very popular. But they use crisp set theory and may not always represent real life problems. It is very difficult to represent constraints of real life optimization problems through crisp constraints. In reality, a small violation of a given constraint is allowed and may generate more efficient solution to the problem. But it cannot happen under crisp environment.

It is known that an optimization problem usually involves objective and constraints. In the last half of century, one paradigm shift in mathematics and consequently in optimization techniques is the invention of fuzzy set theory by L.A. Zadeh (1965). In optimization problem under fuzzy environment, objective(s), constraints are represented by fuzzy sets. Fuzzy optimization techniques are more flexible and generate solutions that are more adequate to real life. There exist several techniques in literature to solve optimization problems under fuzzy environment. Bellman and Zadeh (1970) introduced decision making in a fuzzy environment. The first researchers to publish an operationalization of Bellman and Zadeh's proposed approach to 'Decision making in a fuzzy environment' were H. Tanaka and K. Asai (1973) that was published in English in 1974. Zimmermann (1976, 1978) further developed mathematical model of optimization problems under fuzzy environment. Verdegay (1982) presented another historical paper on fuzzy mathematical programming. Sakawa (1987) developed a comprehensive and interactive multiple objective optimization technique under fuzzy environment. Recently Jimenez and Bilbao (2009) suggested that fuzzy-efficient solution of multiple objective linear

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programming problems (MOLPP) may not be Pareto-optimal in case that one of fuzzy goals is fully achieved. Their proposed procedure extended two-phase approach of Guu, Wu (1997, 1999) and approach of Dubois, Fortemps (1999) to attain Pareto-optimal solutions to MOLPP under fuzzy environment.

But according to Yan-Kuen Wu et al (2015), proposed approach by Jimenez and Bilbao (2009) cannot guarantee to be one general procedure to attain Pareto-optimal solutions. In fact one of the poorly studied problems in the field of fuzzy set theory is its membership function itself. Yan-Kuen Wu et al (2015) redefined membership functions of fuzzy set theory. But that definition of redefined membership functions and usage in mathematical model as well as in given numerical examples in their paper are not analogous. Clearly parts of redefined membership functions where values of objective functions do not exceed sum of goals and tolerances (for minimization type of objective functions) are used in mathematical model and in given numerical examples. But these are not added to set of existing constraints. Thus in problem formulation, redefined membership functions are strictly monotonic where as they are not so in definition. Only upper bounds of membership functions are removed but there still exist lower bounds of membership functions at zero. And in many cases, if the lower bounds of membership functions remain intact, the problem may become infeasible. Membership functions of fuzzy sets provide satisficing result when extreme ends of imprecise information can be quantified within boundary of zero and one. But it may not be always correct to quantify imprecise information within one bounded subset of real line.

Consider fuzzy set $\tilde{A}$ of all players who may win next US Open grand slam (tennis) tournament. Most of us agree with the prediction that Young Donald [current ATP ranking 52] will not be the champion. So it can be safely stated that Young Donald is in $\tilde{A}$ with membership value 0. Again chance of Paul Cayre's [current ATP rank 1000] winning is also nil. So Paul Cayre is also in $\tilde{A}$ with membership value 0. It means that Young Donald and Paul Cayre are at same level! But it is certain that even if Young Donald wins, unfortunately Paul Cayre probably will not participate and hence will not be the winner. Membership function of fuzzy set theory fails to explain this case! An element in fuzzy set may lie partly or never lie or must lie in that set. This concept of lying, partly lying or not lying is well measured by membership functions of fuzzy set theory. But it does not suit in case of must lying (fuzzy membership values are always one), and never lying (fuzzy membership values are always zero).

On the other hand, fuzzy set theory has been widely developed and several new modifications as well as generalizations have appeared. One of these is concept of intuitionistic fuzzy (IF) sets, introduced by K. T. Atanassov (1986). Here not only the degrees of acceptance but also degrees of rejection of elements in a set are considered so that sum of these values does not exceed unity. Analogous to membership function of fuzzy set theory, poorly studied problems in the field of intuitionistic fuzzy set are its membership function and non-membership function. Next Plamen Angelov (1997) introduced optimization technique under IF environment by using degree of acceptance along with degree of rejection, as a natural extension of optimization technique under fuzzy environment. And in literature, the framework of an optimization problem under IF environment is usually considered as the following model:

To maximize the degree(s) of acceptance(s) of intuitionistic fuzzy objective(s) and constraints and to minimize the degree(s) of rejection(s) of intuitionistic fuzzy objective(s) and constraints

$$\underset{x}{Max} \left\{ \mu_i(x) \right\}, x \in R^n, i = 1, \dots, p + q \quad (1)$$

$$\underset{x}{Min} \left\{ \nu_i(x) \right\}, x \in R^n, i = 1, \dots, p + q$$

subject to

$$\nu_i(x) \geq 0, i = 1, \dots, p + q, \quad \mu_i(x) \geq \nu_i(x), i = 1, \dots, p + q,$$

$$\mu_i(x) + \nu_i(x) \leq 1, i = 1, \dots, p + q, \quad x \in X.$$

Where x denotes decision variables, $\mu_i(x)$ denotes the degree of membership of x to i^{th} IF set $z_i(x), i = 1, \dots, p + q$, $\nu_i(x)$ denotes the degree of non-membership of x to i^{th} IF set $z_i(x), i = 1, \dots, p + q$, p denotes the number of objectives, q denotes the number of constraints, $X = \{x \in R^n : Ax \leq b, x \geq 0\}, b = (b_1, b_2, \dots, b_m) \in R^m$, A is an $m \times n$ matrix and R denotes the set of real numbers. Due to symmetry, objective function may be treated as constraint and vice versa.

The rest of the paper is organized as follows: Sect. 2 introduces definitions of $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions. Sect. 3 highlights loopholes of existing technique and suggests remedy to find

optimal solutions to IFO problems. Sect. 4 proposes an interactive and comprehensive algorithm to find optimal solutions under IF environment. Sect. 5 introduces a model on production-distribution planning in supply chain management under IF environment. Sect. 6 shows through numerical examples how existing technique fails and proposed algorithm presents optimal solutions. Sect. 6 also presents two examples where standard membership and non-membership functions have not been used. Sect. 7 gives concluding remarks.

2 Definitions

It is observed that in case of optimization technique under IF environment, one interesting and useful property of membership functions is that higher values of membership functions always give better result of objective functions; and analogous property of non-membership functions is that lower values of non-membership functions give better result of objective functions. *And classical membership functions and non-membership functions are not the only candidates having these amazing characteristics.* Hence functions viz. $T^{(+)}$ -characteristic functions in place of membership functions and $T^{(-)}$ -characteristic functions in place of non-membership functions are defined under IF environment.

Definition 1

Let S denotes the universal set and $\tilde{A}^i$ be an IF subset of S . Then $T^{(+)}$ -characteristic function is denoted by $T_{\tilde{A}^i}^{(+)}(x)$ and is defined as $T_{\tilde{A}^i}^{(+)} : S \rightarrow R$ that assigns a real number $T_{\tilde{A}^i}^{(+)}(x)$ to each element $x \in S$, here $T_{\tilde{A}^i}^{(+)}(x)$ represents degree of membership or acceptance level of $x \in S$ in $\tilde{A}^i$. And $T^{(-)}$ -characteristic function is denoted by $T_{\tilde{A}^i}^{(-)}(x)$ and is defined as $T_{\tilde{A}^i}^{(-)} : S \rightarrow R$ that assigns a real number $T_{\tilde{A}^i}^{(-)}(x)$ to each element $x \in S$, here $T_{\tilde{A}^i}^{(-)}(x)$ represents degree of non-membership or rejection level of $x \in S$ in $\tilde{A}^i$.

3 Obtaining optimal solution to IFO problems

IF optimization (IFO) problems are two stages process that include aggregation of objective(s) and constraints and then defuzzification. Applying the approach by P. Angelov (1997), analogous to fuzzy optimization technique, IFO problem can be transformed into the following problem

$$\begin{aligned} & \max \quad \alpha - \beta \\ & \text{subject to} \\ & \mu_i \geq \alpha, i = 1 \dots k, \quad \nu_i \leq \beta, i = 1 \dots k, \\ & \alpha \geq \beta, \alpha + \beta \leq 1, \beta \geq 0, x \in X. \end{aligned} \tag{2}$$

where $\alpha = \min_{i=1}^{p+q} \mu_i(x)$, denoting minimal degree of acceptance and $\beta = \max_{i=1}^{p+q} \nu_i(x)$, denoting maximal degree of rejection of IF objective(s) and constraints and $k = p + q$. It is to be remembered that in problem formulations as well as in numerical computations, strictly monotonic parts of definitions of membership functions $\mu_i(x), i = 1, \dots, k$ and non-membership functions $\nu_i(x), i = 1, \dots, k$ are used. And triangular IF goal to minimization or lesser-equal to type of $z_i(x)$ is usually denoted by $(a_i; b_i, b'_i)$ with $\mu_i(z_i(a_i)) = 1, \nu_i(z_i(a_i)) = 0$, $\mu_i(z_i(b_i)) = 0$ and $\nu_i(z_i(b'_i)) = 1$ with $b_i < b'_i$ as shown in Fig. 1. Without loss of generality, all constraints are assumed to be active here.

Let us concentrate on philosophy of optimization technique under imprecise environment. In IF optimization technique, several operators can be used. Among those, Angelov (1997) has used an operator that is analogous to classical max-additive operator. Here prime intention is maximum up-gradation of the most under privileged one. Consequently the primary objective in mathematical form is to maximize the difference between minimal degree of acceptance and maximal degree of rejection i.e. to maximize $\alpha - \beta$. In Angelov's paper, objective function $\alpha - \beta$ was subjected to the constraints $\alpha \geq \beta, \alpha + \beta \leq 1, \beta \geq 0$. But, it is shown below that some of these constraints hinder $\alpha - \beta$ from attaining values higher than unity. Again, as numerical examples shows later

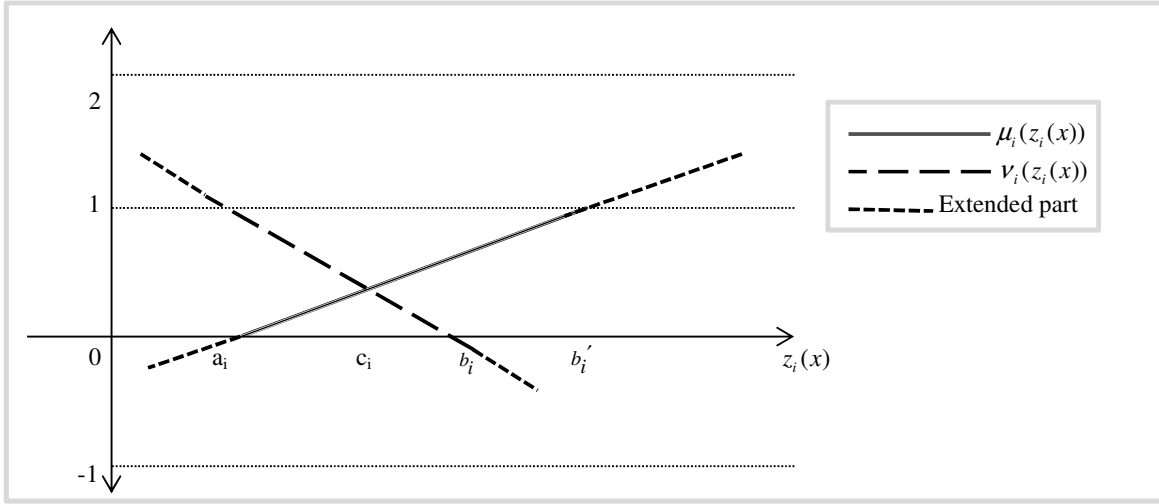


Fig. 1 Strictly monotonically extended membership function and non-membership function (not in scale)

in this paper, some of those constraints may make the problem infeasible. Also, presence of these constraints may yield less acceptable optimal solution to the objective function in some cases. These are now discussed one after another.

First consider a simple linear programming problem (LPP) as follows

$$\begin{aligned} \max \quad & \alpha - \beta \\ \text{subject to} \quad & \\ & \alpha \geq \beta, \alpha + \beta \leq 1, \beta \geq 0. \end{aligned}$$

Here α, β are two decision variables. The optimal solution is $\alpha^* = 1, \beta^* = 0$. '*' denotes optimality.

Clearly, maximum value of objective function $\alpha - \beta$ under given constraints is unity. And $\alpha + \beta \leq 1, \beta \geq 0$ are active constraints in this LPP. Since primary objective is to maximize $\alpha - \beta$ under IF environment, therefore in order to attain higher value than unity of $\alpha - \beta$, the active constraints of this LPP must have to be removed. Again, as numerical example suggests later in this paper, optimal value of $z_i(x)$ cannot be less than a_i when constraints $\alpha + \beta \leq 1, \beta \geq 0$ are present in problem formulation. But, $z_i(x)$ is of minimization or lesser-equal to type and IF goals and/or tolerances are imprecise in nature and any value less than a_i to $z_i(x)$ is more acceptable to decision maker. Hence it is suggested to remove constraints $\alpha + \beta \leq 1, \beta \geq 0$ from model (2).

Again classical membership functions of IF information have upper bound at unity and classical non-membership functions of IF information have lower bound at zero. If constraints $\alpha \leq 1, \beta \geq 0$ are added after removal of $\alpha + \beta \leq 1, \beta \geq 0$, it does not allow maximum value of $\alpha - \beta$ to be higher than unity as well. But, primary objective is to maximize $\alpha - \beta$ under IF environment. Moreover, optimal value of $z_i(x)$ cannot be less than a_i . But, as discussed earlier, $z_i(x)$ is of minimization type and IF goals and/or tolerances are imprecise in nature and any value less than a_i to $z_i(x)$ is more acceptable to decision maker. Hence it is suggested not to add constraints $\alpha \leq 1, \beta \geq 0$ to the model.

On the other hand, constraint $\alpha \geq \beta$ means that minimal degree of acceptance cannot be less than maximum degree of rejection. Consequently, $z_i(x)$ cannot attain values greater than c_i (c_i being abscissa of the point of intersection of membership function and non-membership function of $z_i(x)$). But as IF goals and/or tolerances are imprecise in nature, in many cases, this constraint may make problem infeasible, as shown by numerical example. Hence it is suggested to remove the constraint $\alpha \geq \beta$, if required, so that optimal solution may be found.

Again constraint $\alpha \geq 0$ does not allow objective function $z_i(x)$ to take value greater than b_i . But as IF goals and/or tolerances are imprecise in nature, it may make the problem infeasible, as shown by numerical example. Hence it is suggested to remove constraint $\alpha \geq 0$, if required, so that optimal solution may be found. Hence it is

more useful to have simply strictly monotonic functions viz. $T^{(+)}$ -characteristic functions, in place of membership functions in IFO technique.

Again classical definition of non-membership functions of IF set restricts upper bound at unity. If this constraint $\beta \leq 1$ be added, it may make problem infeasible, as shown by numerical example. Therefore it is more useful to have strictly monotonic functions viz. $T^{(-)}$ -characteristic functions, in place of non-membership functions in IFO technique.

4 General method to solve IFO problem

The above ideas can be further integrated into a general framework and an algorithm may be developed to obtain optimal solution to optimization problem under IF environment. The steps of the proposed algorithm are synthesized as follows

- Step 1. Convert linguistic information to objective data and construct single objective optimization problem where objective and / or constraint(s) may be IF in nature. Ask decision maker to specify goal(s) and tolerance(s) for IF objective and constraint(s).
- Step 2. Define suitable $T^{(+)}$ -characteristic function in such a way that higher value of $T^{(+)}$ -characteristic function always gives better result for IF objective and / or constraint(s). And simultaneously define suitable $T^{(-)}$ -characteristic function in such a way that lower value of $T^{(-)}$ -characteristic function gives better result for objective and / or constraint(s). It is noted that well-defined $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions are always finite.
- Step 3. Construct IF optimization problem as

$$\begin{aligned} & \max \quad \alpha - \beta \\ & \text{subject to} \\ & T_i^+(z_i(x)) \geq \alpha, i = 1, \dots, k, T_i^-(z_i(x)) \leq \beta, i = 1, \dots, k, \\ & \alpha \geq \beta, \beta \geq 0, x \in X. \end{aligned} \quad (3)$$

- Step 4. Solve model (3). If it has optimal solution, go to step 8. Otherwise go to step 5.
- Step 5. If model (3) is infeasible, remove constraint $\alpha \geq \beta$ and add constraint $\alpha \geq 0$. The modified model becomes

$$\begin{aligned} & \max \quad \alpha - \beta \\ & \text{subject to} \\ & T_i^+(z_i(x)) \geq \alpha, i = 1, \dots, k, T_i^-(z_i(x)) \leq \beta, i = 1, \dots, k, \\ & \alpha \geq 0, \beta \geq 0, x \in X. \end{aligned} \quad (4)$$

Solve this modified model. If it has optimal solution, go to step 8. Otherwise, go to step 6.

- Step 6. Remove the newly added constraint $\alpha \geq 0$ and add the constraint $\beta \leq 1$ in last modified model. The modified model becomes

$$\begin{aligned} & \max \quad \alpha - \beta \\ & \text{subject to} \\ & T_i^+(z_i(x)) \geq \alpha, i = 1, \dots, k, T_i^-(z_i(x)) \leq \beta, i = 1, \dots, k, \\ & \beta \leq 1, \alpha \text{ is unrestricted in sign}, x \in X. \end{aligned} \quad (5)$$

Solve it. If it has optimal solution, go to step 8. Otherwise, remove constraint $\beta \leq 1$ and solve. If it has optimal solution, go to step 8. Otherwise the model is infeasible, even under IF environment; go to step 7.

- Step 7. Ask decision maker to modify goal(s) and tolerance(s) for IF objective and constraint(s). Go to step 2. It is noted that goal(s) and/or tolerance(s) of IF objectives should better lie within individual maximum and minimum values.
- Step 8. This solution is optimal to single objective optimization problem under IF environment with specified goals(s) and tolerance(s). Supply this optimal solution to the decision maker. If he/she is satisfied with this solution, stop. Otherwise go to Step 7.

Consequently the amount of product that is produced in each period at least should satisfy the backorder of the previous time period. It leads to the constraint $Q_n \geq B_{n-1}, \forall n$.

And each distribution center for each time period has a minimum requirement of supply that should be satisfied. Hence it leads to the constraint $\sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \forall j, \forall n$.

In real life, capacity of man-hour is imprecise in nature. And the usual limitations in the capacity of machine-hour, man-hour and the available warehouse volume of the factory are given by the constraints

$$U * Q_n \leq F_n, \forall n, \quad V * Q_n \leq M_n, \forall n, \quad \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \forall n.$$

Again the decision maker (DM) can only look for one contractor to dispatch the products to distribution centers. Hence the constraint is obtained as $\sum_{k=1}^K Y_k = 1$. The non-negativity restrictions on decision variables and special restrictions further lead to the constraints $Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \forall n$ and $Y_k = \{0,1\}$. Here it may be assumed that the objective function has imprecise aspiration level.

Then the mathematical model under IF environment is as follows

$$\text{Min } z \approx \sum_{n=1}^N (CP_n * Q_n + CI_n * I_n + CB_n * B_n + \sum_{k=1}^K \sum_{j=1}^J Y_k (CT_{jkn} * T_{jkn} + FC_{jkn})) * (1 + MARR)^n \quad (6)$$

subject to

$$C_{1n} : I_{n-1} + Q_n - I_n = \sum_{k=1}^K \sum_{j=1}^J (Y_k * T_{jkn}), \forall n, \quad C_{2n} : I_{n-1} - B_{n-1} + Q_n - I_n + B_n = D_n, \forall n,$$

$$C_{3n} : Q_n \geq B_{n-1}, \forall n, \quad C_{4jn} : \sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \forall j, \forall n,$$

$$C_{5n} : U * Q_n \leq F_n, \forall n, \quad C_{6n} : V * Q_n \leq M_n, \forall n, \quad C_{7n} : \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \forall n,$$

$$C_8 : \sum_{k=1}^K Y_k = 1, \quad C_9 : Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \forall n, \quad C_{10} : Y_k = \{0,1\}.$$

This IF mathematical model is flexible in the value of objective functions and also has the vagueness in some constraints. Hence the task is to convert it as model (1) and next as model (2).

6 Application of production-distribution model under IF environment

IF mathematical model is now applied on this packaged-drinking water bottling factory. Here the data are taken analogous to Ariaifar et al (2014). The production planner of the factory has to provide an overall perspective regarding the amount of production, inventory and also backordering of their main product for a midterm scheduling period, as an aggregate production plan. There are two alternative contractors and three distribution centers. The data related to bottling factory are as follows

- The main product is a bottle of packaged drinking water. Each dozen of bottles is packed in one box.
- The planning horizon of the factory is three months June, July and August.
- The initial inventory is 500 boxes and the end inventory at August is zero.
- The initial and end back order volume is zero.
- Production of each box utilizes 0.015 machine hours of the capacity of the factory.
- The required man power to produce one box of the product equals 0.09 man-hours. The dimension of each box is 31x27x37 cm³. In 1 m³ of the space of the warehouse, 32 boxes can be stored. The dimension of the warehouse is 15x20 m² and height is 1.5m. Hence the volume of the warehouse is 450 m³.
- The escalating factor of the cost is 8%.

The other data are given in the following tables.

Table 1. Forecasted data on demand and factory capacity

Period	Demand (Box) (D)	Factory capacity (machine-hour) (F)
June	12400	193
July	12000	185
August	12500	196

Table 2. Minimum supply for each destination (box)

Period	City A	City B	City C
June	2000	2500	5000
July	1500	2000	6000
August	1000	2000	6500

Table 3. Costs data (\$/box)

Period	Production cost	Inventory holding cost	Back ordering cost
June	1.60	0.10	0.50
July	1.80	0.08	0.40
August	1.25	0.09	0.35

Table 4. Per box transportation cost for each destination (\$/box)

Period	City A		City B		City C	
	K=1	K=2	K=1	K=2	K=1	K=2
June	0.094	0.112	0.106	0.142	0.064	0.082
July	0.096	0.118	0.102	0.144	0.060	0.080
August	0.092	0.112	0.105	0.156	0.058	0.088

Table 5. Fixed transportation cost for each destination (\$)

Period	City A		City B		City C	
	K=1	K=2	K=1	K=2	K=1	K=2
June	110	50	180	80	100	30
July	120	55	175	75	80	25
August	130	60	160	70	90	35

Table 6. Imprecise data on maximum manpower available in period n (man-hour)

Period	Maximum manpower in period n (man-hour) (M)
June	Availability is 1110 man-hour. It can go up to 1115 man-hour but will not exceed 1118 man-hour.
July	Availability is 1095 man-hour. It can go up to 1100 man-hour but will not exceed 1104 man-hour.
August	Availability is 1097 man-hour. It can go up to 1100 man-hour but will not exceed 1102 man-hour.

Clearly, here $J = 3, N = 3$ and $K = 2$.

6.1.1. Numerical Example 1 (removal of constraints $\alpha + \beta \leq 1, \beta \geq 0$)

In this example, it is shown that the constraints $\alpha + \beta \leq 1, \beta \geq 0$ may hinder the primary objective $\alpha - \beta$ from attaining values higher than unity.

Suppose that the decision maker chooses goal for the objective function z at \$71,000 and decides that cost may increase up to \$75,000 but in no case it exceeds \$76,000.

Also, the constraints C_{61}, C_{62}, C_{63} are IF in nature, as given in Table 6. Based on those information, membership and non-membership functions of the IF objective and constraints are constructed as follows

$$\mu_1(z) = \begin{cases} 1, & \text{if } z \leq 71000 \\ \frac{75000 - z}{4000}, & \text{if } 71000 \leq z \leq 75000 \\ 0, & \text{if } z \geq 75000 \end{cases} \quad \nu_1(z) = \begin{cases} 0, & \text{if } z \leq 71000 \\ \frac{z - 71000}{5000}, & \text{if } 71000 \leq z \leq 76000 \\ 1, & \text{if } z \geq 76000 \end{cases}$$

Table 7. Imprecise data on maximum manpower available in period n (man-hour)

Period	Maximum manpower in period n (man-hour) (M)
June	Availability is 1110 man-hour. It can go up to 1115 man-hour but will not exceed 1118 man-hour.
July	Availability is 1095 man-hour. It can go up to 1100 man-hour but will not exceed 1104 man-hour.
August	Availability is 1077 man-hour. It can go up to 1080 man-hour but will not exceed 1082 man-hour.

$$\mu_2(0.09*Q_1) = \begin{cases} 1, & \text{if } 0.09*Q_1 \leq 1110 \\ \frac{1115-0.09*Q_1}{5}, & \text{if } 1110 \leq 0.09*Q_1 \leq 1115 \\ 0, & \text{if } 0.09*Q_1 \geq 1115 \end{cases} \quad \nu_2(0.09*Q_1) = \begin{cases} 0, & \text{if } 0.09*Q_1 \leq 1110 \\ \frac{0.09*Q_1-1110}{8}, & \text{if } 1110 \leq 0.09*Q_1 \leq 1118 \\ 1, & \text{if } 0.09*Q_1 \geq 1118 \end{cases}$$

$$\mu_3(0.09*Q_2) = \begin{cases} 1, & \text{if } 0.09*Q_2 \leq 1095 \\ \frac{1100-0.09*Q_2}{5}, & \text{if } 1095 \leq 0.09*Q_2 \leq 1100 \\ 0, & \text{if } 0.09*Q_2 \geq 1100 \end{cases} \quad \nu_3(0.09*Q_2) = \begin{cases} 0, & \text{if } 0.09*Q_2 \leq 1095 \\ \frac{0.09*Q_2-1095}{9}, & \text{if } 1095 \leq 0.09*Q_2 \leq 1104 \\ 1, & \text{if } 0.09*Q_2 \geq 1104 \end{cases}$$

$$\mu_4(0.09*Q_3) = \begin{cases} 1, & \text{if } 0.09*Q_3 \leq 1097 \\ \frac{1100-0.09*Q_3}{3}, & \text{if } 1097 \leq 0.09*Q_3 \leq 1100 \\ 0, & \text{if } 0.09*Q_3 \geq 1100 \end{cases} \quad \nu_4(0.09*Q_3) = \begin{cases} 0, & \text{if } 0.09*Q_3 \leq 1097 \\ \frac{0.09*Q_3-1097}{5}, & \text{if } 1097 \leq 0.09*Q_3 \leq 1102 \\ 1, & \text{if } 0.09*Q_3 \geq 1102 \end{cases}$$

Then the single objective optimization problem is obtained as follows

$$\begin{aligned} & \max \quad \alpha - \beta & (7) \\ & \text{subject to} \\ & \mu_i \geq \alpha, \forall i = 1, 2, 3, 4, \quad \nu_i \leq \beta, \forall i = 1, 2, 3, 4, \quad \alpha + \beta \leq 1, \quad \alpha \geq \beta, \beta \geq 0, \\ & C_{1n} : I_{n-1} + Q_n - I_n = \sum_{k=1}^K \sum_{j=1}^J (Y_k * T_{jkn}), \quad \forall n, \quad C_{2n} : I_{n-1} - B_{n-1} + Q_n - I_n + B_n = D_n, \quad \forall n, \\ & C_{3n} : Q_n \geq B_{n-1}, \quad \forall n, \quad C_{4jn} : \sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \quad \forall j, \forall n, \\ & C_{5n} : U * Q_n \leq F_n, \quad \forall n, \quad C_{7n} : \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \quad \forall n, \\ & C_8 : \sum_{k=1}^K Y_k = 1, \quad C_9 : Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \quad \forall n, \quad C_{10} : Y_k = \{0, 1\}. \end{aligned}$$

Using Lingo 15.0.32, the optimal solution is obtained as $\alpha^* - \beta^* = 1, z^* = \71000 .

As discussed earlier, if constraints $\alpha + \beta \leq 1, \beta \geq 0$ are removed from problem (7), the new optimal solution is obtained as $\bar{\alpha} - \bar{\beta} = 1.26, \bar{z} = \69975.49 . Since maximizing up-gradation of most misfortunate i.e. maximizing $\alpha - \beta$ is primary objective, this solution is more preferable to decision maker. Moreover, optimal value of objective function is more acceptable after removing constraints $\alpha + \beta \leq 1, \beta \geq 0$. Clearly, usage of proposed algorithm generates more preferable optimal solution to the problem than traditional IFO technique.

6.1.2. Numerical Example 2 (removal of upper bound of degree of acceptance)

After the constraints $\alpha + \beta \leq 1, \beta \geq 0$ are removed, suggestion to add constraint $\alpha \leq 1$ may pop up in order to restrict upper bound of minimal degree of acceptance at unity. The following example shows that such addition is not permissible.

Suppose that the decision maker chooses goal for the objective function z at \$70,800 and decides that cost may increase up to \$75,000 but in no case it exceeds \$78,800.

Again, constraints C_{61}, C_{62}, C_{63} are IF in nature, as given in Table 7. Based on these information, the membership and non-membership functions of the IF objective and constraints are constructed. And finally the single objective optimization problem is obtained as follows

Table 8. Imprecise data on maximum manpower available in period n (man-hour)

Period	Maximum manpower in period n (man-hour) (M)
June	Availability is 1115man-hour. It can go up to 1170 man-hour but will not exceed 1173 man-hour.
July	Availability is 1095 man-hour. It can go up to 1100 man-hour but will not exceed 1110 man-hour.
August	Availability is 1090 man-hour. It can go up to 1100 man-hour but will not exceed 1110 man-hour.

$$\begin{aligned}
& \max \alpha - \beta \tag{8} \\
& \text{subject to} \\
& \mu_i \geq \alpha, \forall i = 1, 2, 3, 4, \quad v_i \leq \beta, \forall i = 1, 2, 3, 4, \quad \alpha \leq 1, \quad \alpha \geq \beta, \beta \geq 0, \\
& C_{1n} : I_{n-1} + Q_n - I_n = \sum_{k=1}^K \sum_{j=1}^J (Y_k * T_{jkn}), \forall n, \quad C_{2n} : I_{n-1} - B_{n-1} + Q_n - I_n + B_n = D_n, \forall n, \\
& C_{3n} : Q_n \geq B_{n-1}, \forall n, \quad C_{4jn} : \sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \forall j, \forall n, \\
& C_{5n} : U * Q_n \leq F_n, \forall n, \quad C_{7n} : \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \forall n, \\
& C_8 : \sum_{k=1}^K Y_k = 1, \quad C_9 : Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \forall n, \quad C_{10} : Y_k = \{0, 1\}.
\end{aligned}$$

Using Lingo 15.0.32, the optimal solution is obtained as $\alpha^* - \beta^* = 1, z^* = \70800 . As discussed earlier, if constraints $\alpha \leq 1, \beta \geq 0$ are not added, the new optimal solution is obtained as $\bar{\alpha} - \bar{\beta} = 1.16, \bar{z} = \70109.63 .

Since maximizing up-gradation of most misfortunate is prime target i.e. maximizing $\alpha - \beta$ is primary objective, this solution is more preferable to decision maker. Moreover, optimal value of objective function is more acceptable when constraints $\alpha \leq 1, \beta \geq 0$ are not added. Clearly, usage of proposed algorithm generates more preferable optimal solution to the problem than traditional IFO technique.

6.1.3. Numerical Example 3 (removal of constraint $\alpha \geq \beta$)

This example shows that the constraint $\alpha \geq \beta$ may make a problem infeasible whose optimal solution may exist with $\alpha^* < \beta^*$.

Suppose that the decision maker chooses goal for the objective function z at \$69,500 and decides that cost may increase up to \$70,000 but in no case it exceeds \$70,300.

Again, constraints C_{61}, C_{62}, C_{63} are IF in nature, as given in Table 8. Based on these information, the membership and non-membership functions of the IF objective and constraints are constructed. And finally the single objective optimization problem is obtained as follows

$$\begin{aligned}
& \max \alpha - \beta \tag{9} \\
& \text{subject to} \\
& \mu_i \geq \alpha, \forall i = 1, 2, 3, 4, \quad v_i \leq \beta, \forall i = 1, 2, 3, 4, \quad \alpha \geq \beta, \\
& C_{1n} : I_{n-1} + Q_n - I_n = \sum_{k=1}^K \sum_{j=1}^J (Y_k * T_{jkn}), \forall n, \quad C_{2n} : I_{n-1} - B_{n-1} + Q_n - I_n + B_n = D_n, \forall n, \\
& C_{3n} : Q_n \geq B_{n-1}, \forall n, \quad C_{4jn} : \sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \forall j, \forall n, \\
& C_{5n} : U * Q_n \leq F_n, \forall n, \quad C_{7n} : \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \forall n, \\
& C_8 : \sum_{k=1}^K Y_k = 1, \quad C_9 : Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \forall n, \quad C_{10} : Y_k = \{0, 1\}.
\end{aligned}$$

Using Lingo 15.0.32, the problem has no feasible solution.

If constraint $\alpha \geq \beta$ is removed, then the optimal solution is obtained as $\alpha^* = 0.14, \beta^* = 0.53, \alpha^* - \beta^* = -0.39, z^* = \$69,928.49$.

Clearly, removal of the constraint $\alpha \geq \beta$ on problem formulation and usage of proposed algorithm generates optimal solution to the problem where traditional IFO technique fails.

Table 9. Imprecise data on maximum manpower available in period n (man-hour)

Period	Maximum manpower in period n (man-hour) (M)
June	Availability is 1100 man-hour. It can go up to 1105 man-hour but will not exceed 1108 man-hour.
July	Availability is 1105 man-hour. It can go up to 1110 man-hour but will not exceed 1111 man-hour.
August	Availability is 1077 man-hour. It can go up to 1080 man-hour but will not exceed 1082 man-hour.

Table 10. Imprecise data on maximum manpower available in period n (man-hour)

Period	Maximum manpower in period n (man-hour) (M)
June	Availability is 1075 man-hour. It can go up to 1080 man-hour but will not exceed 1083 man-hour.
July	Availability is 1105 man-hour. It can go up to 1110 man-hour but will not exceed 1112 man-hour.
August	Availability is 1077 man-hour. It can go up to 1080 man-hour but will not exceed 1082 man-hour.

6.1.4. Numerical Example 4(removal of lower bound of degree of acceptance)

After the constraint $\alpha \geq \beta$ has been removed, suggestion to add constraint $\alpha \geq 0$ may pop up in order to restrict lower bound of minimal degree of acceptance at zero. The following example shows that such addition of constraint is not permissible.

Suppose that the decision maker chooses goal for the objective function z at \$69,500 and decides that cost may increase up to \$70,000 but in no case it exceeds \$70,300.

Again, constraints C_{61}, C_{62}, C_{63} are IF in nature, as given in Table 9. Based on these information, the membership and non-membership functions of the IF objective and constraints are constructed. And finally the single objective optimization problem is obtained as follows

$$\begin{aligned}
 & \max \alpha - \beta & (10) \\
 & \text{subject to} \\
 & \mu_i \geq \alpha, \forall i = 1, 2, 3, 4, \quad \nu_i \leq \beta, \forall i = 1, 2, 3, 4, \quad \alpha \geq 0, \\
 & C_{1n} : I_{n-1} + Q_n - I_n = \sum_{k=1}^K \sum_{j=1}^J (Y_k * T_{jkn}), \forall n, \quad C_{2n} : I_{n-1} - B_{n-1} + Q_n - I_n + B_n = D_n, \forall n, \\
 & C_{3n} : Q_n \geq B_{n-1}, \forall n, \quad C_{4jn} : \sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \forall j, \forall n, \\
 & C_{5n} : U * Q_n \leq F_n, \forall n, \quad C_{7n} : \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \forall n, \\
 & C_8 : \sum_{k=1}^K Y_k = 1, \quad C_9 : Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \forall n, \quad C_{10} : Y_k \in \{0, 1\}.
 \end{aligned}$$

Using Lingo 15.0.32, the problem has no feasible solution. As discussed earlier, if constraint $\alpha \geq 0$ is not added, then the optimal solution is obtained as $\alpha^* = -0.18, \beta^* = 0.74, \alpha^* - \beta^* = -0.92, z^* = \$70,091.42$.

Clearly, proposed algorithm generates optimal solution to the problem whereas restriction on lower bound of minimal degree of acceptance makes the problem infeasible. Therefore removal of lower bound of degree of acceptance of IF information is proposed.

6.1.5. Numerical Example 5(removal of upper bound of degree of rejection)

After the constraint $\alpha \geq \beta$ has been removed and constraint $\alpha \geq 0$ is not added, suggestion to add $\beta \leq 1$ may pop up in order to restrict upper bound of maximal degree of rejection at unity. The following example shows that such addition of constraint is not permissible.

Suppose that the decision maker chooses goal for the objective function z at \$69,500 and decides that cost may increase up to \$70,000 but in no case it exceeds \$70,300.

Again, constraints C_{61}, C_{62}, C_{63} are IF in nature, as given in Table 10. Based on these information, the membership and non-membership functions of the IF objective and constraints are constructed. And finally the single objective optimization problem is obtained as follows

Table 11. Imprecise data on maximum manpower available in period n (man-hour)

Period	Maximum manpower in period n (man-hour) (M)
June	Availability is 1075 man-hour. It can go up to 1080 man-hour but will not exceed 1083 man-hour.
July	Availability is 1100 man-hour. It can go up to 1105 man-hour but will not exceed 1107 man-hour.
August	Availability is 1072 man-hour. It can go up to 1080 man-hour but will not exceed 1082 man-hour.

$$\begin{aligned}
& \max \alpha - \beta \quad (11) \\
& \text{subject to} \\
& \mu_i \geq \alpha, \forall i = 1, 2, 3, 4, \quad \nu_i \leq \beta, \forall i = 1, 2, 3, 4, \quad \beta \leq 1, \\
& C_{1n} : I_{n-1} + Q_n - I_n = \sum_{k=1}^K \sum_{j=1}^J (Y_k * T_{jkn}), \forall n, \quad C_{2n} : I_{n-1} - B_{n-1} + Q_n - I_n + B_n = D_n, \forall n, \\
& C_{3n} : Q_n \geq B_{n-1}, \forall n, \quad C_{4jn} : \sum_{k=1}^K (Y_k * T_{jkn}) \geq R_{jn}, \forall j, \forall n, \\
& C_{5n} : U * Q_n \leq F_n, \forall n, \quad C_{7n} : \sum_{k=1}^K \sum_{j=1}^J Y_k (S * T_{jkn}) \leq AW_n, \forall n, \\
& C_8 : \sum_{k=1}^K Y_k = 1, \quad C_9 : Q_n, O_n, I_n, B_n, T_{jkn} \geq 0 \forall n, \quad C_{10} : Y_k = \{0, 1\}.
\end{aligned}$$

Using Lingo 15.0.32, the problem has no feasible solution. As discussed earlier, if constraint $\beta \leq 1$ is not added, then the optimal solution is obtained as $\alpha^* = -0.75, \beta^* = 1.09, \alpha^* - \beta^* = -1.84, z^* = \$70,371.21$.

Clearly, proposed algorithm generates optimal solution to the problem whereas restriction on upper bound of maximal degree of rejection makes the problem infeasible. Therefore removal of upper bound of degree of rejection of IF information is proposed.

6.2.1. Numerical example using proposed algorithm with first set of $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions

Step 1. Suppose that the decision maker chooses goal for the objective function z at \$69,500 and decides that cost may increase up to \$70,000 but in no case it exceeds \$70,300. Again, constraints C_{61}, C_{62}, C_{63} are IF in nature, as given in Table 11.

Step 2. Based on these information, $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions are constructed as follows

$$\begin{aligned}
T^{(+)}_1(z) &= \frac{70000 - z}{500} \forall z & T^{(-)}_1(z) &= \frac{z - 69500}{800} \forall z \\
T^{(+)}_2(0.09 * Q_1) &= \frac{1080 - 0.09 * Q_1}{5} \forall Q_1 & T^{(-)}_2(0.09 * Q_1) &= \frac{0.09 * Q_1 - 1075}{8} \forall Q_1 \\
T^{(+)}_3(0.09 * Q_2) &= \frac{1105 - 0.09 * Q_2}{5} \forall Q_2 & T^{(-)}_3(0.09 * Q_2) &= \frac{0.09 * Q_2 - 1100}{7} \forall Q_2 \\
T^{(+)}_4(0.09 * Q_3) &= \frac{1080 - 0.09 * Q_3}{8} \forall Q_3 & T^{(-)}_4(0.09 * Q_3) &= \frac{0.09 * Q_3 - 1072}{10} \forall Q_3
\end{aligned}$$

Step 3. Then the single objective optimization problem under IF environment is obtained as follows

$$\begin{aligned}
& \max \alpha - \beta \\
& \text{subject to} \\
& T_i^+(z_i(x)) \geq \alpha, i = 1, 2, 3, 4, \quad T_i^-(z_i(x)) \leq \beta, i = 1, 2, 3, 4, \quad \alpha \geq \beta, \beta \geq 0, x \in X.
\end{aligned}$$

Step 4. Using Lingo 15.0.32, the problem has no feasible solution.

Step 5. Next remove constraint $\alpha \geq \beta$ and add constraint $\alpha \geq 0$ and solve the modified model. Using Lingo 15.0.32, this problem also has no feasible solution.

Step 6. Next remove constraint $\alpha \geq 0$ and add constraint $\beta \leq 1$. Solve this modified model by using Lingo 15.0.32. It has no feasible solution. As discussed earlier, remove constraint $\beta \leq 1$ and solve. Using Lingo 15.0.32, the optimal solution is obtained as $\alpha^* = -0.69, \beta^* = 1.21, \alpha^* - \beta^* = -1.90, z^* = \$70,345.83$.

Step 8. This solution is optimal to the problem under IF environment for given goals and tolerances. Supply this solution to the decision maker. If he/she is satisfied with this solution, stop. Otherwise, request the decision maker to choose different goals and tolerances for IF objective and constraints.

Hence proposed algorithm generates optimal solution to the problem under imprecise environment where as traditional IFO technique fails.

6.2.2. Numerical Example using proposed algorithm with second set of $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions

Suppose decision maker is unable to choose goals and tolerances for imprecise objective and constraints. In that case, strictly monotonically decreasing $T^{(+)}$ -characteristic functions and strictly monotonically increasing $T^{(-)}$ -characteristic functions can be used for minimization and lesser equal to type of objectives and constraints. Consider the following example.

Step 1. Suppose that the decision maker is unable to choose goals and tolerances for imprecise objective z and constraints C_{61}, C_{62}, C_{63} .

Step 2. Construct strictly monotonically decreasing $T^{(+)}$ -characteristic functions and strictly monotonically increasing $T^{(-)}$ -characteristic functions of IF objective and constraints as follows

$$\begin{aligned} T^{(+)}_1(z) &= -z \quad \forall z, & T^{(-)}(z) &= z \quad \forall z, & T^{(+)}_2(0.09*Q_1) &= -0.09*Q_1 \quad \forall Q_1, \\ T^{(-)}_2(0.09*Q_1) &= 0.09*Q_1 \quad \forall Q_1, & T^{(+)}_3(0.09*Q_2) &= -0.09*Q_2 \quad \forall Q_2, & T^{(-)}_3(0.09*Q_2) &= 0.09*Q_2 \quad \forall Q_2, \\ T^{(+)}_4(0.09*Q_3) &= -0.09*Q_3 \quad \forall Q_3, & T^{(-)}_4(0.09*Q_3) &= 0.09*Q_3 \quad \forall Q_3. \end{aligned}$$

Step 3. Then the single objective optimization problem under IF environment is obtained as follows

$$\max \quad \alpha - \beta$$

subject to

$$T_i^+(z_i(x)) \geq \alpha, i = 1, 2, 3, 4, T_i^-(z_i(x)) \leq \beta, i = 1, 2, 3, 4, \alpha \geq \beta, \beta \geq 0, x \in X.$$

Step 4. Using Lingo 15.0.32, the optimal solution is obtained as $z^* = \$68,565.85$.

Step 8. This is the optimal solution to the IFO problem.

Here completely new set of $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions have been used for imprecise objective and constraints. Several other such $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions may be constructed. These can produce optimal solutions to the IFO problems.

7 Conclusion

In this paper, an algorithm is developed to find optimal solutions for optimization problems under IF environment. IF technique is one of the richest apparatus for formulation of optimization problems under imprecise environment, thereby generating more satisfying result than crisp optimization technique. Philosophically the prime objective of optimization problem is maximum up-gradation of most under-privileged imprecise information, i.e. in mathematical form the objective is to maximize the difference between minimal degree of acceptance and maximal degree of rejection.

In this paper it is shown how constraints present in traditional well established technique hinders objective function from attaining values higher than unity. Again, constraint that minimal degree of acceptance must not be less than maximal degree of rejection may make a problem infeasible etc.

Therefore it is proposed to use strictly monotonic $T^{(+)}$ -characteristic function in place of membership function and strictly monotonic $T^{(-)}$ -characteristic function in place of non-membership function in IFO technique. This algorithm is designed in such a way that it always generates optimal solution under IF environment, whenever crisp feasible space is non-null.

In this paper, proposed algorithm is applied on a production distribution planning in supply chain management. The information supplied being imprecise in nature, IF optimization technique is used to find optimal solution. The proposed algorithm generates optimal solution to this model whereas traditional IFO technique fails for given goals and tolerances. Moreover it is interactive in nature. Hence scope of further improvement is always present in it.

It may further be concluded that, under these circumstances, the issue of getting no feasible solution under IF environment yielded from several published methods seems worthwhile or even necessary to reconsider.

Moreover, a decision maker may not be present always to supply goal(s) and/or tolerance(s). He/she may not be well equipped as well. Moreover, it may be sometimes difficult to construct classical membership and/or non-membership functions. Usage of strictly monotonic $T^{(+)}$ -characteristic functions and strictly monotonic $T^{(-)}$ -characteristic functions helps in these cases also. Many such $T^{(+)}$ -characteristic functions and $T^{(-)}$ -characteristic functions can be constructed as necessary to generate optimal solution under imprecise environment.

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